HW 1-1 (Suggested solutions posted on 02/01/2017) Name:

HW 1.1. (Sample spaces, Events, Union, Intersection, and Complement.)

Consider an experiment that selects a cell phone camera and records the recycle time of a flash (the time taken to ready the camera for another flash). The possible values for this time depend on the resolution of the timer and on the minimum and maximum recycle times. It is known that all recycle times are between 1 and 10 seconds. Assume that the recycle times are no less than 1.5 and no more than 5 when a cell phone camera equipped with battery A; between 2 and 8 when a cell phone camera equipped with battery B.

(1) What is the sample space S of recycle times?

(2) What is the event E_1 of recycle times when camera equipped with battery A?

(3) What is the event E_2 of recycle times when camera equipped with battery B?

(4) $E_1 \cup E_2 =$

(5) $E_1 \cap E_2 =$

(6) $E_1' =$

(7) $E_1 \cup E_2' =$

Bonus!

$((E_1 \cap E_2)')' =$

(1) $S = (1, 10)$,

(2) $E_1 = [1.5, 5]$,

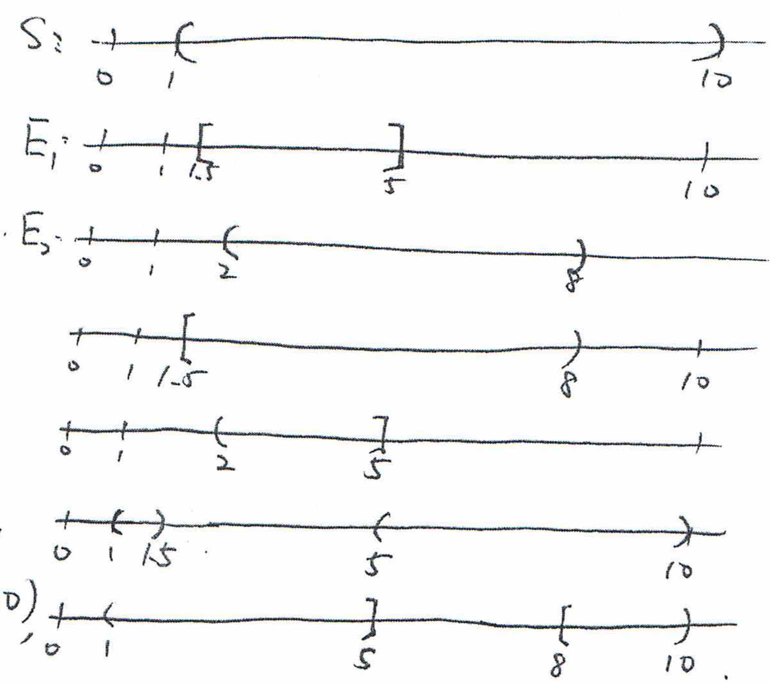
(3) $E_2 = (2, 8)$,

(4) $E_1 \cup E_2 = [1.5, 8)$,

(5) $E_1 \cap E_2 = (2, 5]$,

(6) $E_1' = (1, 1.5) \cup (5, 10)$,

(7) $E_1 \cup E_2' = (1, 5] \cup [8, 10)$,



Bonus! $((E_1 \cap E_2)')' = E_1 \cap E_2 = (2, 5]$.

Name:

HW 1.2. (Probability, Conditional Probability, and Independence.) Samples of emissions from three suppliers are classified for conformance to air-quality specifications. The results from 100 samples are summarized as follows:

			Conforms				
			B Yes		B'	No	
Supplier	A	1	22	22	8	8	30
	A'	2	25	55	15	5	70
		3	30			10	
			77			23	100

Let A denote the event that a sample is from supplier 1, and let B denote the event that a sample conforms to specifications. If a sample is selected at random, determine the following probabilities:

1. $P(A) =$ $30/100 = 0.3$
2. $P(B) =$ $77/100 = 0.77$
3. $P(A') =$ $70/100 = 0.7$
4. $P(A \cap B) =$ $22/100 = 0.22$
5. $P(A \cup B) =$ $P(A)+P(B)-P(A\cap B) = 0.3+0.77-0.22 = 0.85$
6. $P(A \cup B') =$ $P(A)+P(B')-P(A\cap B') = 0.3+0.23-8/100 = 0.45$
7. $P(A'|B) =$ $P(A'\cap B)/P(B) = (55/100)/(77/100) = \frac{5}{7}$

Bonus! Are events A and B independent? Why?

$P(A)P(B) = (0.3)(0.77) = 0.231$

$P(A\cap B) = 0.22$

Since $P(A)P(B) \neq P(A\cap B)$, A and B are not independent!

HW 1.3. (Probability and Conditional Probability.) An article in the *The Canadian Entomologist* reported on the life of the alfalfa weevil from eggs to adulthood. The following table shows the number of larvae that survived at each stage of development from eggs to adults.

Eggs	Early Larvae	Late Larvae	Pre-pupae	Late Pupae	Adults
421	412	306	45	35	31

1. What is the probability an egg survives to adulthood?

2. What is the probability of survival to adulthood given survival to the late larvae stage?

3. What stage has the lowest probability of survival to the next stage?

Bonus! What is the probability of larvae cannot survive to late larvae given it survived at early larvae?

(1). P("an egg survives to adulthood")

$$= \frac{\text{number of Adults}}{\text{number of Eggs}} = \frac{31}{421} \approx 0.0736.$$

(2). P("survival to adulthood" | "survival to late larvae")

$$= \frac{P(\text{"survival to adulthood and late larvae"})}{P(\text{"Survival to late larvae"})}$$

$$= \frac{P(\text{"survival to adulthood"})}{P(\text{"survival to late larvae"})}$$

$$= \frac{\text{number of Adults}}{\text{number of Late Larvae}} = \frac{31}{306} \approx 0.1013.$$

(3) Eggs: $\frac{412}{421} \approx 0.9786$.

Early Larvae: $\frac{306}{412} \approx 0.7427$.

Late Larvae: $\frac{45}{306} \approx 0.1471$.

Prepupae: $\frac{35}{45} \approx 0.7778$.

Late pupae: $\frac{31}{35} \approx 0.8857$.

So Late Larvae has the lowest probability of survial to the next stage.

Bonus: $1 - \frac{306}{412} \approx 0.2573$.

HW 1.4. (Equally Likely Outcomes, Conditional Probability, and Independence.) A computer system uses passwords that contain exactly six characters, and each character is one of the 26 lowercase letters (a–z) or 26 uppercase letters (A–Z) or 10 integers (0–9). Let S denote the set of all possible password, and let A and B denote the events that consist of passwords with only letters or only integers, respectively. Suppose that all passwords in S are **equally likely**.

1. How many outcomes in sample space S?
2. How many outcomes in event A?
3. How many outcomes in event B?
4. How many outcomes in event $A \cap B$?
5. $P(A) =$
6. $P(A|B) =$
7. Are events A and B independent? Why?

Bonus! P(password contains exactly 2 integers given that it contains at least 1 integer)

(1) $(26+26+10)^6 = 62^6$

(2) $(26+26)^6 = 52^6$

(3) 10^6

(4) $A \cap B = \phi$. no outcome in event $A \cap B$.

(5) $P(A) = \dfrac{\text{number of outcomes in } A}{\text{number of outcomes in } S} = \dfrac{52^6}{62^6}$

(6) Since $A \cap B = \phi$, then $P(A\cap B) = 0$

$P(B) = \dfrac{\text{number of outcomes in } B}{\text{number of outcomes in } S} = \dfrac{10^6}{62^6}$

$P(A|B) = P(A\cap B)/P(B) = 0/\left(\frac{10^6}{62^6}\right) = 0$

(7) Since $0 = P(A|B) \neq P(A) = \frac{52^6}{62^6}$, then A and B are not independant.

Bonus! $P(\text{2 integers and at least 1 integers}) = P(\text{2 integers}) = \dfrac{\binom{6}{2}10^2\cdot\binom{4}{4}52^4}{62^6}$

$P(\text{at least 1 integers}) = 1 - P(\text{0 integers}) = 1 - \dfrac{52^6}{62^6}$

$P(\text{2 integers} \mid \text{1 integers}) = \left[\binom{6}{2}10^2\binom{4}{4}52^4/62^6\right] \Big/ \left[1 - \frac{52^6}{62^6}\right] \approx 0.2962.$

HW 1.5. (Conditional Probability and Total Probability Rules.) When coded messages are sent, there are sometimes errors in transmisssion. In particular, Morse code use "dots" and "dashes," which are known to occur in the proportion of 3 : 4. This means that for any given symbol,

$$P(\text{dot sent}) = \frac{3}{7} \quad \text{and} \quad P(\text{dash sent}) = \frac{4}{7}.$$

Suppose there is interference on the transmission line, and with probability $\frac{1}{8}$ a dot is mistakenly received as a dash, and vice versa. Please determine the following probabilities:

1. $P(\text{dash received} \mid \text{dot sent}) =$ $1/8$
2. $P(\text{dot received} \mid \text{dot sent}) =$ $1 - P(\text{dash received}) = 1 - 1/8 = 7/8$
3. $P(\text{dot received} \mid \text{dash sent}) =$ $1/8$
4. $P(\text{dot received}) =$

Bonus! $P(\text{dot sent} \mid \text{dot received})$.

4. $P(\text{dot received}) = P(\text{dot received} \cap \text{dot sent}) + P(\text{dot received} \cap \text{dash sent})$

(Law of total probability) $\overset{\text{?}}{=} P(\text{dot received} \mid \text{dot sent})\,P(\text{dot sent}) + P(\text{dot received} \mid \text{dash sent})\,P(\text{dash sent})$

$= (7/8)(3/7) + (1/8)(4/7) = \frac{21}{56} + \frac{4}{56} = \frac{25}{56}$.

Bonus! $P(\text{dot sent} \mid \text{dot received})$

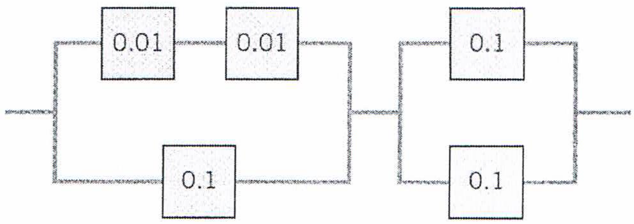
$= P(\text{dot sent} \cap \text{dot received}) / P(\text{dot received})$

$= P(\text{dot received} \cap \text{dot sent}) / P(\text{dot received})$

$= [P(\text{dot received} \mid \text{dot sent})\,P(\text{dot sent})] / P(\text{dot received})$

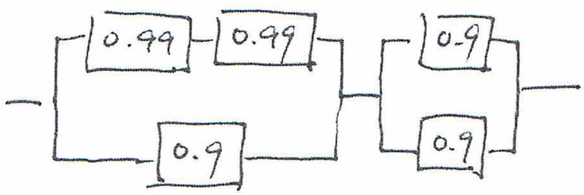
$= [(7/8)(3/7)] / \left(\frac{25}{56}\right) = \left(\frac{21}{56}\right) / \left(\frac{25}{56}\right) = \frac{21}{25} = 0.84$.

HW 1.6. (Independence and Advanced Circuit.) The following circuit operates if and only if there is a path of functional devices from left to right. Assume devices fail independently and that the probability of failure of each device is as shown. What is the probability that the circuit operates?

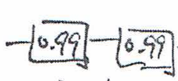


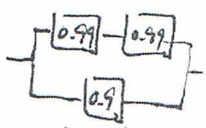
Bonus! What is the most important device in this circuit? Please explain why you think so.

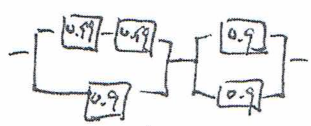
The probability of "function" of each device is as shown:



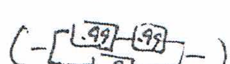
Note that devices fail independently!

P( works) $= (0.99)^2$; P( works) $= 1-(1-0.9)(1-0.9)$;

P( works) $= 1-(1-(0.99)^2)(1-0.9)$;

P( works) $= [1-(1-(0.99)^2)(1-0.9)][1-(1-0.9)(1-0.9)]$

≈ 0.9880.



Bonus! Since the first part has low failure rate, so I believe the devices in the second part () is more important.

HW 1.7. (Pmf, Cdf, Mean, and Variance.) Given the following pmf,

x	-2	-1	0	1	2
$f(x)$	0.2	0.4	0.1	0.2	0.1

1. Find the following probabilities

$$(a)\ P(X \le 2) \qquad (b)\ P(X > -2)$$
$$(c)\ P(-1 \le X \le 1) \qquad (d)\ P(X \le -1 \text{ or } X = 2)$$
$$(e)\ P(-1 \le X < 1) \qquad (f)\ P(-1 < X < 1)$$

2. Find the cdf.

3. Plot the cdf.

4. $E(X) =$

5. $V(X) =$

Bonus!

$$(a)\ P(X^2 \le 3) \quad (b)\ P(X^2 > 2) \quad (c)\ P(0 < X^2 \le 1)$$

1. (a) $P(X \le 2) = 1$ (b) $P(X > -2) = 1 - P(X \le -2)$

$= 1 - P(X=-2) = 1 - f(-2) = 1 - 0.2 = 0.8$

(c) $P(-1 \le X \le 1) = P(X=-1) + P(X=0) + P(X=1)$

$= f(-1) + f(0) + f(1) = 0.4 + 0.1 + 0.2 = 0.7$

(d) $P(X \le -1 \text{ or } X = 2) = P(X=-2) + P(X=-1) + P(X=2)$

$= f(-2) + f(-1) + f(2) = 0.2 + 0.4 + 0.1 = 0.7$

(e) $P(-1 \le X < 1) = P(X=-1) + P(X=0)$

$= f(-1) + f(0) = 0.4 + 0.1 = 0.5$

(f) $P(-1 < X < 1) = P(X=0) = f(0) = 0.1$

2. $F(x) = \begin{cases} 0 & , \quad x < 0 \\ 0.2 & , \quad 0 \le x < 1 \\ 0.6 & , \quad 1 \le x < 2 \\ 0.7 & , \quad 2 \le x < 3 \\ 0.9 & , \quad 3 \le x < 4 \\ 1 & , \quad 4 \le x . \end{cases}$

4.

x	-2	-1	0	1	2
$f(x)$	0.2	0.4	0.1	0.2	0.1
x^2	4	1	0	1	4

$$E(X) = \sum x f(x) = -2(0.2) + (-1)(0.4) + 0(0.1) + 1(0.2) + 2(0.1) = -0.4$$

5.

$$E(X^2) = \sum x^2 f(x) = 4(0.2) + 1(0.4) + 0(0.1) + 1(0.2) + 4(0.1) = 1.8$$

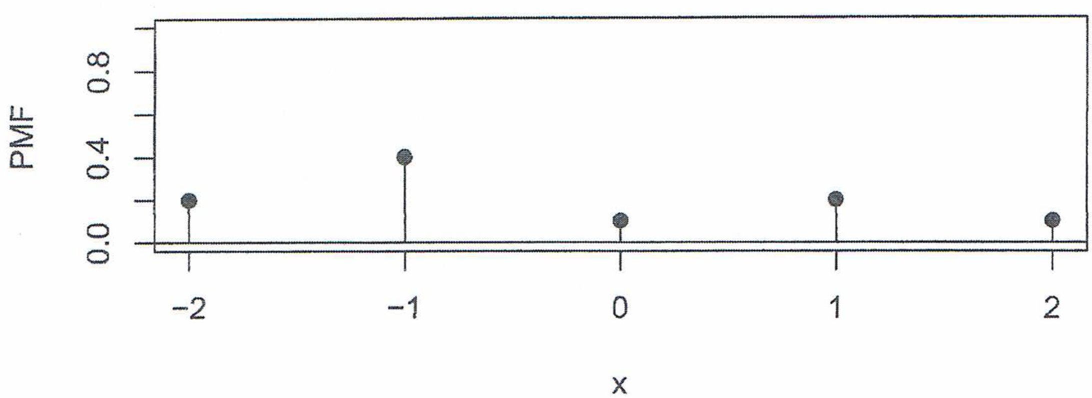
$$V(X) = E(X^2) - [E(X)]^2 = 1.8 - (-0.4)^2 = 1.64.$$

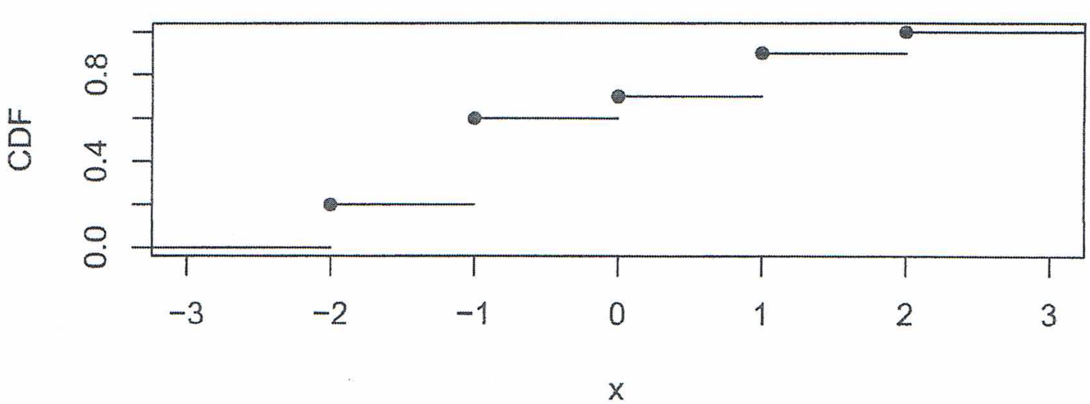
Bonus! (a) $P(X^2 \le 3) = P(-\sqrt{3} \le X \le \sqrt{3}) = P(X=-1) + P(X=0) + P(X=1)$
$= 0.4 + 0.1 + 0.2 = 0.7.$

(b) $P(X^2 > 2) = P(X > \sqrt{2} \text{ or } X < -\sqrt{2}) = P(X=-2) + P(X=2)$
$= 0.2 + 0.1 = 0.3.$

(c) $P(0 < X^2 \le 1) = P(-1 \le X < 0 \text{ or } 0 < X \le 1) = P(X=-1) + P(X=1)$
$= 0.4 + 0.2 = 0.6.$

3.





HW 1.8. (Pmf, Cdf, Mean, and Variance.) Errors in an experimental transmission channel are found when the transmission is checked by a certifier that detects missing pulses. The number of errors found in an eight-bit byte is a random variable with the following distribution:

$$F(x) = \begin{cases} 0 & x < 1 \\ 0.7 & 1 \le x < 4 \\ 0.9 & 4 \le x < 7 \\ 1 & 7 \le x \end{cases}$$

1. Determine each of the following probabilities:

   (a) $P(X \le 4)$ (b) $P(X > 7)$
   (c) $P(X < 2)$ (d) $P(1 < X \le 7)$
   (e) $P(1 \le X \le 7)$

2. Find the pmf.

3. Plot the pmf.

4. $E(X) =$

5. $V(X) =$

Bonus!

(a) $P(X \le 4.6)$ (b) $P(X > 7.2)$ (c) $P(1.2 < X \le 7.5)$

1. (a) $P(X \le 4) = F(4) = 0.9$

(b) $P(X > 7) = 1 - P(X \le 7) = 1 - F(7) = 1 - 1 = 0$

(c) $P(X < 2) = P(X \le 2)$ (no probability assigned at $X=2$)

$= F(2) = 0.7$

(d) $P(1 < X \le 7) = F(7) - F(1) = 1 - 0.7 = 0.3$

(e) $P(1 \le X \le 7) = P(X=1) + P(1 < X \le 7)$

Since the hight of jump at $X=1$ is 0.7 which is the probability that assigned at $X=1$, then $P(X=1) = 0.7$.

So $P(1 \le X \le 7) = 0.7 + 0.3 = 1$.

2. According to the hight of jumps of cdf $F(x)$, we have pmf function only assigning positive values at 1, 4, and 7. In other words:

$$f(x) = P(X=x) = \begin{cases} 0.7, & x=1 \\ 0.2, & x=4 \\ 0.1, & x=7 \\ 0, & \text{o.w.} \end{cases}$$

4.

x	1	4	7
$f(x)$	0.7	0.2	0.1
x^2	1	16	49

$$E(X) = \sum x f(x) = 1(0.7) + 4(0.2) + 7(0.1) = 2.2.$$

5. $$E(X^2) = \sum x^2 f(x) = 1(0.7) + 16(0.2) + 49(0.1) = 8.8$$

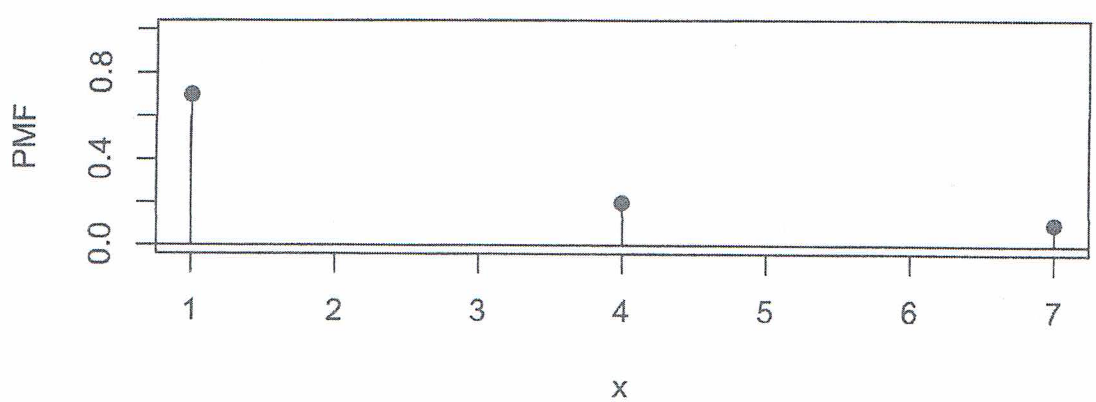
$$V(X) = E(X^2) - [E(X)]^2 = 8.8 - (2.2)^2 = 3.96.$$

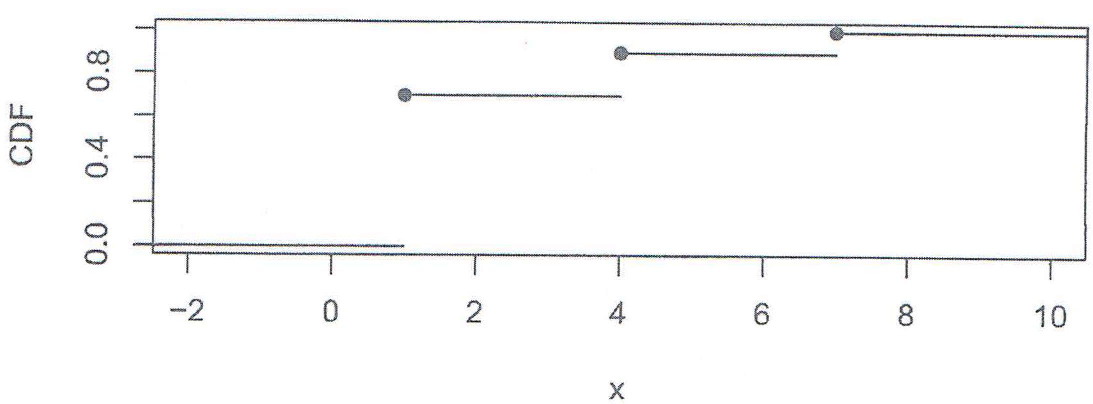
Bonus! (a) $P(X \le 4.6) = F(4.6) = 0.9$

(b) $P(X > 7.2) = 1 - P(X \le 7.2) = 1 - F(7.2) = 1 - 1 = 0$

(c) $P(1.2 < X \le 7.5) = F(7.5) - F(1.2) = 1 - 0.7 = 0.3.$

3.





HW 1.9. **(Mean and Variance.)** There is a chance that a bit transmitted through a digital transmission channel is received in error. Let X equal the number of bits in error in the next four bits transmitted. The possible values for X are $\{0, 1, 2, 3, 4\}$. Based on a model for the errors that is presented in the following section, probabilities for these values will be determined.

$$P(X = 0) = 0.6561, \quad P(X = 1) = 0.2916,$$
$$P(X = 2) = 0.0486, \quad P(X = 3) = 0.0036,$$
$$P(X = 4) = 0.0001.$$

The probability distribution of X is specified by the possible values along with the probability of each. Then the pmf of X is:

$$f(x) = \begin{cases} 0.6561, & \text{if } x = 0 \\ 0.2916, & \text{if } x = 1 \\ 0.0486, & \text{if } x = 2 \\ 0.0036, & \text{if } x = 3 \\ 0.0001, & \text{if } x = 4. \end{cases}$$

1. $E(X) =$
2. $V(X) =$
3. $E(10X + 5) =$
4. $V(10X + 5) =$
5. $E(\sqrt{X}) =$

Bonus! $V(\sqrt{X}) =$

x	0	1	2	3	4
$f(x)$	0.6561	0.2916	0.0486	0.0036	0.0001
x^2	0	1	4	9	16
$\sqrt{x}$	0	1	$\sqrt{2}$	$\sqrt{3}$	2

1. $E(X) = \sum x f(x) = 0(0.6561) + 1(0.2916) + 2(0.0486)$
$+ 3(0.0036) + 4(0.0001) = 0.4$

2. $E(X^2) = \sum x^2 f(x) = 0(0.6561) + 1(0.2916) + 4(0.0486)$
$+ 9(0.0036) + 16(0.0001) = 0.52.$

$V(X) = E(X^2) - [E(X)]^2 = 0.52 - (0.4)^2 = 0.52 - 0.16 = 0.36$

3. $E(10X+5) = 10 \cdot E(X) + 5 = 10(0.4) + 5 = 4 + 5 = 9$

4. $V(10X+5) = V(10X) = 10^2 V(X) = 100(0.36) = 36$

5. $E(\sqrt{X}) = \sum \sqrt{x} f(x) = 0(0.6561) + 1(0.2916) + \sqrt{2}(0.0486)$
$+ \sqrt{3}(0.0036) + 2(0.0001) \approx 0.3668$

Bonus! $E((\sqrt{X})^2) = E(X) = 0.4$

$V(\sqrt{X}) = E((\sqrt{X})^2) - [E(\sqrt{X})]^2$

$\approx 0.4 - (0.3668)^2$

$\approx 0.2655.$